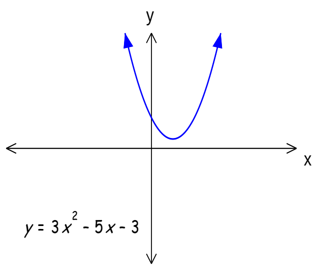
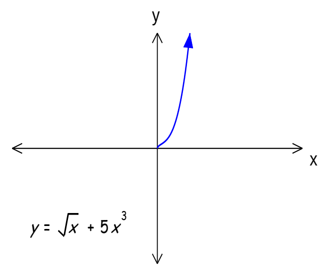
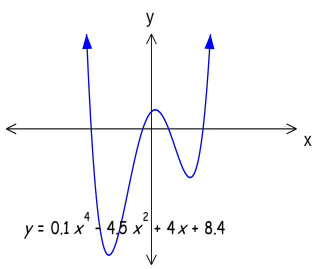
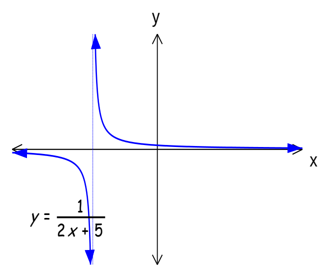
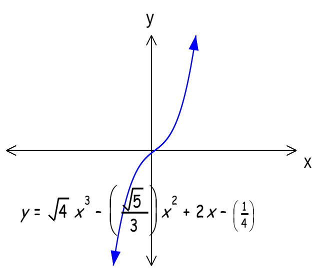
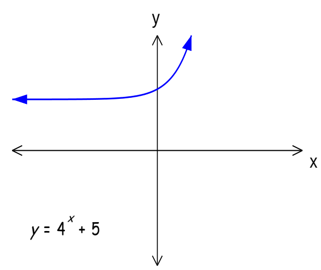
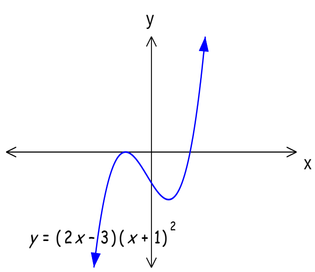
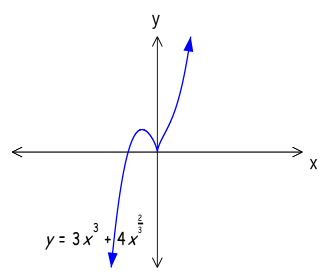
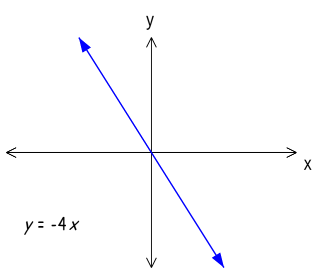
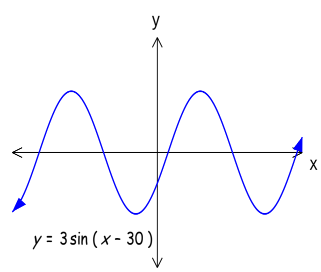
MHF4U – Unit 2: Polynomial Functions (Chapter 3 in book)

<u>Lesson 1: Exploring Polynomial Functions</u>

Look at the following graphs. The graphs on the left are polynomials; those on the right are not. Identify properties of polynomial functions. Be sure to look at both the graphs and the equations.

Polynomials	Not polynomials
 $y = 3x^2 - 5x - 3$	 $y = \sqrt{x} + 5x^3$
 $y = 0.1x^4 - 4.5x^2 + 4x + 8.4$	 $y = \frac{1}{2x+5}$
 $y = \sqrt{4}\,x^3 - \left(\frac{\sqrt{5}}{3}\right)x^2 + 2x - \left(\frac{1}{4}\right)$	 $y = 4^x + 5$
 $y = (2x-3)(x+1)^2$	 $y = 3x^3 + 4x^{\frac{2}{3}}$
 $y = -4x$	 $y = 3\sin(x - 30)$

Characteristics of polynomial functions:

Must have:	**Cannot have:**
Domain: $\{x \in \Re\}$	Asymptotes
Range: may be all real numbers or may have either an upper or lower bound (not both)	Fractional exponents
	Multiple variables
	Trig. Functions, n^{th} root of variable

Equations of polynomial functions:

Linear: $y = mx + b$ or $Ax + By + C = 0$ $\Rightarrow$ degree 1

Quadratic: $y = ax^2 + bx + c$ $\Rightarrow$ degree 2

Cubic: $y = ax^3 + bx^2 + cx + d$ $\Rightarrow$ degree 3

Quartic : $y = ax^4 + bx^3 + cx^2 + dx + e$ $\Rightarrow$ degree 4

Quintic : $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ $\Rightarrow$ degree 5

Note : in each case, $a \neq 0$

General equation of a polynomial function:

$$y = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number.

- a_n is called the leading coefficient
- The degree of a polynomial is the value of the highest exponent
- A polynomial in standard form has descending powers of x.

Finite Differences

Recall: → first differences are constant for a linear function

→ second differences are constant for a quadratic function

Ex. $y = x^3 - 2x^2 + 1$

x	y	1st	2nd	3rd
0	1			
1	0	0 – 1 = -1		
2	1	1 – 0 = 1	2	
3	10	10 – 1 = 9	8	6
4	33	33 – 10 = 23	14	6
5	76	76 – 33 = 43	20	6

This cubic function has constant third differences.

Let's generalize:

Linear function: $y = ax + b$

x	y	1st
0		
1		
2		
3		
4		

HW: Pg. 127 #1, 2, 3bd, 5

FINITE DIFFERENCES

Linear function: $y = ax + b$

x	y	1st
0	b	
1	a + b	a
2	2a + b	a
3	3a + b	a
4	4a + b	a

Quadratic function: $y = ax^2 + bx + c$

x	y	1st	2nd
0	c		
1	a + b + c	a + b	
2	4a + 2b + c	3a + b	2a
3	9a + 3b + c	5a + b	2a
4	16a + 4b + c	7a + b	2a

Cubic function: $y = ax^3 + bx^2 + cx + d$

x	y	1st	2nd	3rd
0	d			
1	a + b + c + d	a + b + c		
2	8a + 4b + 2c + d	7a + 3b + c	6a + 2b	
3	27a + 9b + 3c + d	19a + 5b + c	12a + 2b	6a
4	64a + 16b + 4c + d	37a + 7b + c	18a + 2b	6a
5	125a + 25b + 5c + d	61a + 9b + c	24a + 2b	6a

Quartic function: $y = ax^4 + bx^3 + cx^2 + dx + e$

x	y	1st	2nd	3rd	4th
0	e				
1	a + b + c + d + e	a + b + c + d			
2	16a + 8b + 4d + 2d + e	15a + 7b + 3c + d	14a + 6b + 2c		
3	81a + 27b + 9c + 3d + e	65a + 19b + 5c + d	50a + 12b + 2c	36a + 6b	
4	256a + 64b + 16c + 4d + e	175a + 37b + 7c + d	110a + 18b + 2c	60a + 6b	24a
5	625a + 125b + 25c + 5d + e	369a + 61b + 9c + d	194a + 24b + 2c	84a + 6b	24a
6	1296a + 216b + 36c + 6d + e	671a + 91b + 11c + d	302a + 30b + 2c	108a + 6b	24a

Degree	Constant Difference	Coefficient of Constant Difference
1	a	1
2	2a	$2 = 2 \cdot 1 = 2!$
3	6a	$6 = 3 \cdot 2 \cdot 1 = 3!$
4	24a	$24 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$
…		
n	n!a	n!

Note: 5! means 5 factorial
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

MHF4U – Unit 2: Polynomial Functions (Chapter 3 in book)
Lesson 2: Characteristics of Polynomial Functions

Worksheet…

For a linear function a determines the slope of the line:
- ✓ if $a > 0$ the function is increasing
- ✓ if $a < 0$ the function is decreasing

For a quadratic function:
- ✓ if $a > 0$ the function is decreasing then increasing
- ✓ if $a < 0$ the function is increasing then decreasing

Cubic Function
- ✓ d is the y-intercept
- ✓ if $a > 0$, as x increases f(x) increases (then decreases then increases)
- ✓ if $a < 0$, as x increases f(x) decreases (then increases then decreases)

Quartic Function
- ✓ e is the y-intercept
- ✓ if $a > 0$, as x increases f(x) decreases then increases (then decreases then increases)
- ✓ if $a < 0$, as x increases f(x) increases then decreases (then increases then decreases)

FUNCTION	# ZEROS POSSIBLE	SKETCHES
linear	0, 1 $(\infty \text{ if } y = 0)$	
quadratic	0, 1, 2	
cubic	1, 2, 3	

HW: Pg. 136 #1, 2, 5, 7, 8

T: Pg. 137 #9, 14, 15, 16

Name:________________________

Date: __________

Investigating End Behaviour and Turning Points

Complete the following table for each of the given equations.

a) $f(x) = 9x^2 - 8x - 2$
b) $f(x) = -x^4 - 3x^3 + 3x^2 + 8x + 5$
c) $f(x) = 2x^6 - 13x^4 + 15x^2 + x - 17$
d) $f(x) = -2x^4 - 4x^3 + 3x^2 + 6x + 9$
e) $f(x) = x^3 - 5x^2 + 3x + 4$
f) $f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 - 20$
g) $f(x) = -x^7 + 8x^5 - 16x^3 + 8x$
h) $f(x) = -2x^3 + 8x^2 - 5x + 3$

Function	Degree	# of Turning Points	Sign of Leading Coefficient	Even or Odd Degree?	End Behaviour as $x \to \infty$	End Behaviour as $x \to -\infty$
a						
b						
c						
d						
e						
f						
g						
h						

Make a conjecture about the maximum number of turning points in the graph of a polynomial function with degree 8, 9 or n.

Make a conjecture about the end behaviour of a function with a degree that is
a) even

b) odd

Make a conjecture about the end behaviour of a function with a degree that is
a) even and has a positive leading coefficient

b) even and has a negative leading coefficient

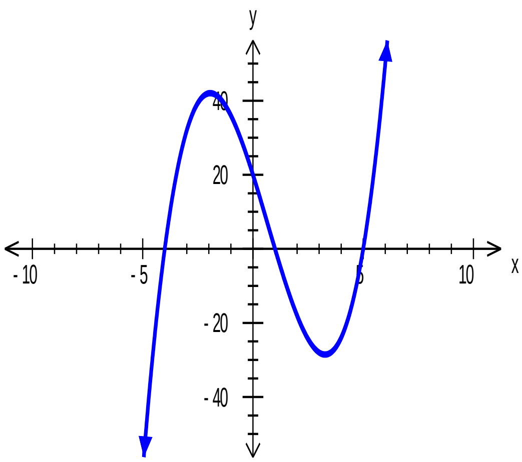
c) odd and has a positive leading coefficient

d) odd and has a negative leading coefficient

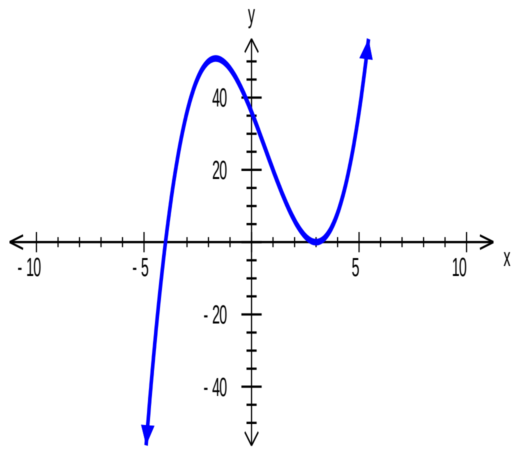
MHF4U – Unit 2: Polynomial Functions (Chapter 3 in book)

<u>Lesson 3: Characteristics of Polynomial Functions in Factored Form</u>

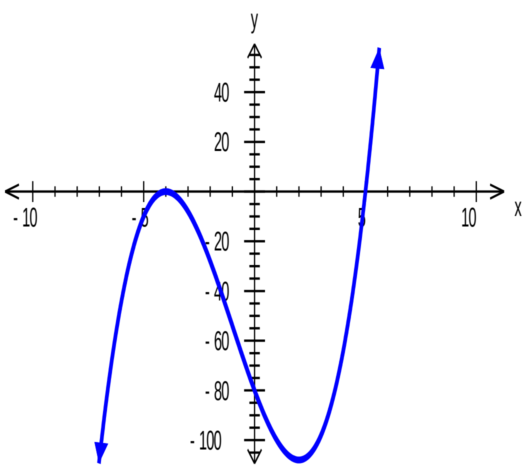
Describe the shape of the graph near each zero.



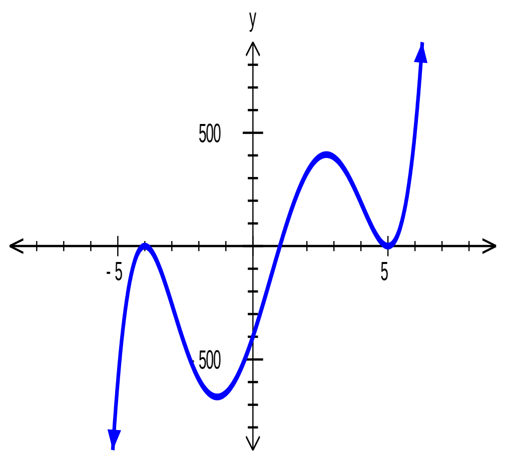
$y = (x-5)(x-1)(x+4)$



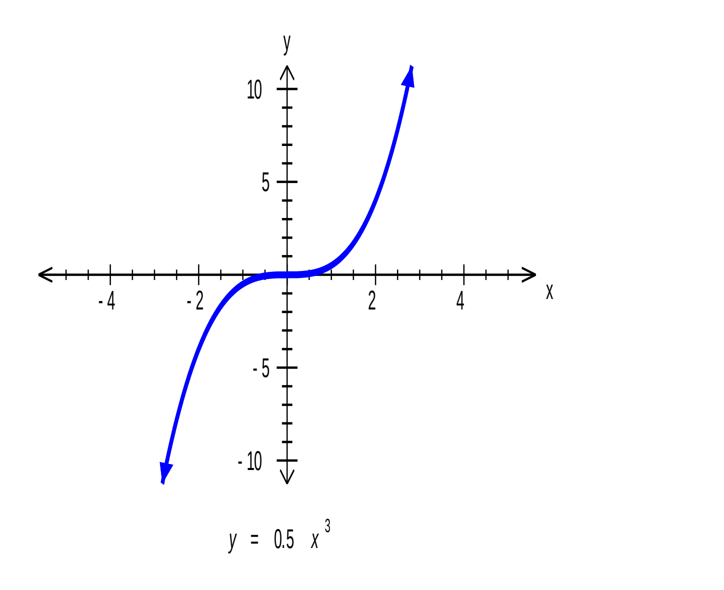
$y = (x-3)^2(x+4)$

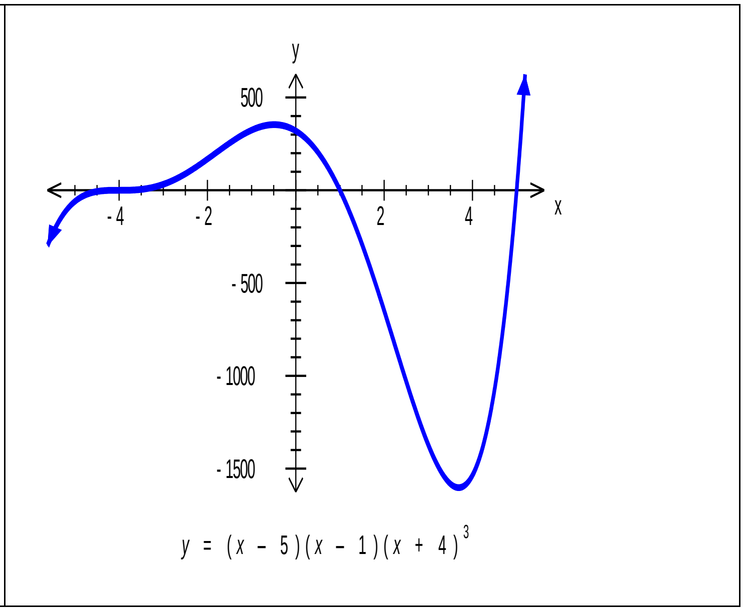


$y = (x-5)(x+4)^2$



$y = (x-5)^2(x-1)(x+4)^2$





<u>Zeros of a Polynomial Function:</u>

TYPE	EQUATION	ZEROS
linear	f(x) = k (x - s)	s
quadratic	f(x) = k (x - s)(x - t)	s, t
cubic	f(x) = k (x - s)(x - t)(x - u)	s, t, u

Changing the value of k will produce a graph in the same **family of functions**, i.e. same zeros but stretched or compressed vertically or reflected about the x-axis.

If any of the factors of a polynomial function are squared → double root (turning point is x-intercept).
Also, the x-axis is **tangent** to the curve at this point.
Tangent line: *line that touches a curve at one point within a small interval.*

If any of the factors of a polynomial function are cubed → there is a horizontal tangent at the x-intercept but the curve crosses the x-axis.

If you know the zeros of a function and one other point you can determine the value of k and therefore the equation of the function.

ex. Determine the equation, in standard form, of the cubic function having zeros at -2, -1 and 1 and y-intercept at 10.

$$
\begin{aligned}
y &= k(x-s)(x-t)(x-u) \\
y &= k(x+2)(x+1)(x-1) \\
10 &= k(0+2)(0+1)(0-1) \\
10 &= -2k \\
k &= -5 \\
y &= -5(x+2)(x+1)(x-1) \\
y &= -5(x^3+2x^2-x-2) \\
y &= -5x^3-10x^2+5x+10
\end{aligned}
$$

Remember:

- The equation of a cubic function with only one zero: $y = k(x - s)^3$
- The equation of a cubic function with exactly two zeros

  $y = k(x - s)^2 (x - t)$

  or $y = k(x - s) (x - t)^2$

Order: the exponent to which each factor is raised.

HW: Pg. 146 #1, 4, 5, [7a i,ii,iii, 8bcd → no equations], 13b

T: Pg. 148 #14 (no graph, don't rewrite in factored form), 15

MHF4U – Unit 2: Polynomial Functions (Chapter 3 in book)
Lesson 4: Transformations of Cubic and Quartic Functions

TI- Nspire file…

Parent functions for cubic & quartic:

$y = x^3$	
x	y
-2	
-1	
0	
1	
2	

$y = x^4$	
x	y
-2	
-1	
0	
1	
2	

Transformations apply to cubic and quartic functions in the same way they do to all other functions.

Ex. The graph of $y = x^3$ is transformed to obtain the graph of $y = -3[2(x+1)]^3 + 5$.

a) State the parameters and describe the corresponding transformations.
b) Complete the table

$y = x^3$	$y = (2x)^3$	$y = -3(2x)^3$	$y = -3[2(x+1)]^3 + 5$
(-2, -8)			
(-1, -1)			
(0, 0)			
(1, 1)			
(2, 8)			

c) Sketch a graph of $y = -3[2(x+1)]^3 + 5$

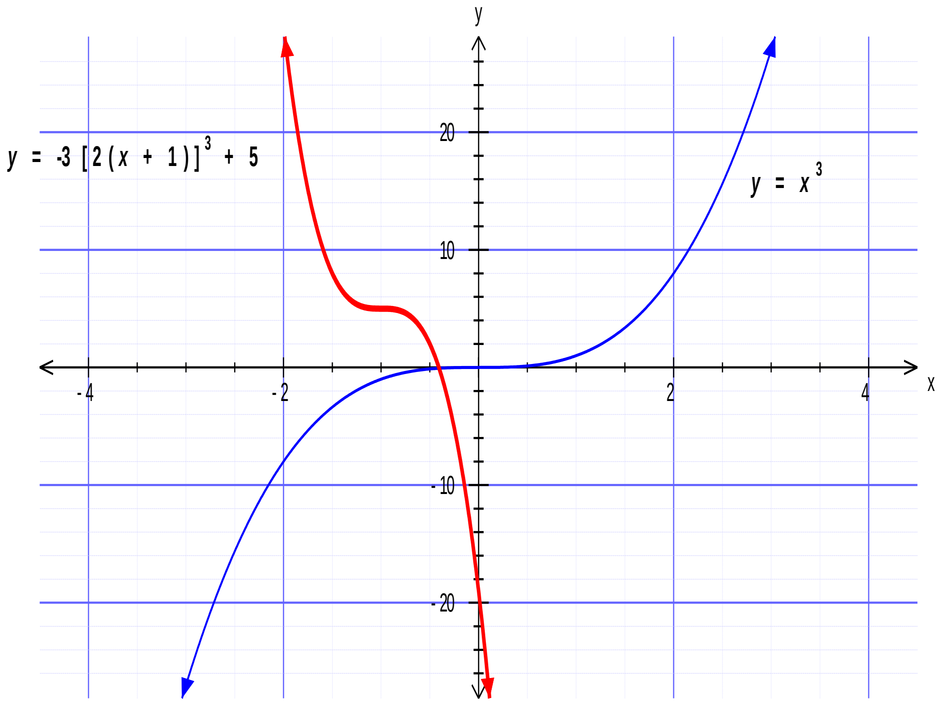
Solution:

a) k = 2 → horizontal compression by a factor of 2*
→ divide x-coordinates by 2
a = -3 → vertical stretch by a factor of 3 and reflection about the x-axis
→ multiply y-coordinates by -3
d = -1 → horizontal translation 1 unit left
→ subtract 1 from x-coordinates
c = 5 → vertical translation up 5 units
→ add 5 to y-coordinates

b)

$y = x^3$	$y = (2x)^3$	$y = -3(2x)^3$	$y = -3[2(x+1)]^3 + 5$
(-2, -8)	(-1, -8)	(-1, 24)	(-2, 29)
(-1, -1)	(-0.5, -1)	(-0.5, 3)	(-1.5, 8)
(0, 0)	(0, 0)	(0, 0)	(-1, 5)
(1, 1)	(0.5, 1)	(0.5, -3)	(-0.5, 2)
(2, 8)	(1,8)	(1, -24)	(0, -19)

c)



Ex. Determine the roots of $y = -\frac{1}{4}(x+5)^3 - 16$.

Let $y = 0$

$$0 = -\frac{1}{4}(x+5)^3 - 16$$
$$16 = -\frac{1}{4}(x+5)^3$$
$$-64 = (x+5)^3$$
$$\sqrt[3]{-64} = x+5$$
$$-4 = x+5$$
$$x = -9$$

Don't forget ± when taking an even root.

HW: Pg. 155 #1, 3ab, 6, 8, 9a

T: Pg. 157 #7, 10, 13

MHF4U – Unit 2: Polynomial Functions (Chapter 3 in book)
Lesson 5: Remainder & Factor Theorems – Part I

Ex. Sketch $y = x^3 - 6x^2 - 7x$

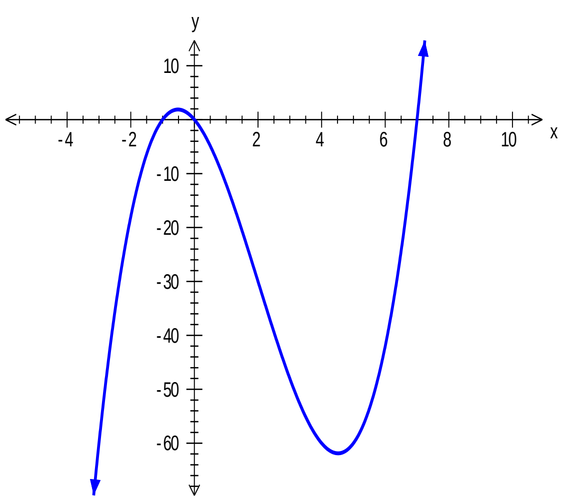
We know:
- ✓ Cubic function
- ✓ Positive leading coefficient therefore
  - as $x \to \infty, y \to \infty$
  - as $x \to -\infty, y \to -\infty$

In order to sketch this we need to know the zeros.
Let $y = 0$, then factor

$$x^3 - 6x^2 - 7x = 0$$
$$x\left(x^2 - 6x - 7\right) = 0$$
$$x(x-7)(x+1) = 0$$

Zeros at -1, 0 and 7.

In order to approximate the local maximum and minimum values, substitute x-values in between the zeros.
If $x = -0.5$, $y = 1.875$
If $x = 3$, $y = -48$
If $x = 4$, $y = -60$
If $x = 5$, $y = -60$



Graphing polynomials is easier if they are written in factored form. How do we factor higher order polynomials?

Let's look at properties of factors.

Numerically:

$$30 = 2 \cdot 15$$
$$= 2 \cdot 3 \cdot 5$$

$30 \div 2 = 15$
$30 \div 3 = 10$
$30 \div 5 = 6$
$30 \div 4 = 7$ *remainder* 2
$30 \div 9 = 3$ *remainder* 3

If you divide 30 by any of its factors, the remainder is 0. This is also true of polynomials.

$x^2 + 5x + 6 = (x+2)(x+3)$

$(x^2 + 5x + 6) \div (x+2) = (x+3)$
$(x^2 + 5x + 6) \div (x+3) = (x+2)$
$(x^2 + 5x + 6) \div (x+1) = (x+4)$ *remainder* 2

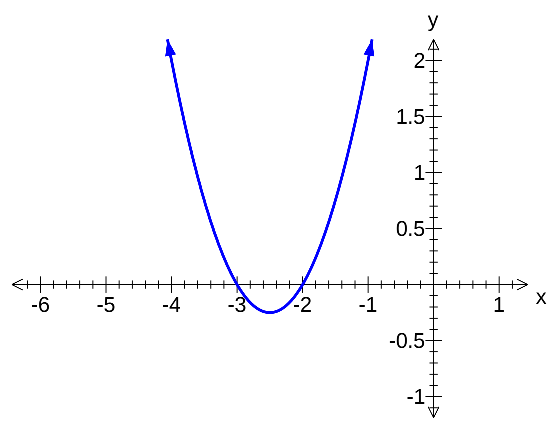
Therefore if a binomial goes into a polynomial with a remainder of 0, it is a factor.

How can we check this without dividing?

The factored form of a polynomial allows you to easily find the roots.

Ex. $f(x) = x^2 + 5x + 6$
In factored form: $f(x) = (x+2)(x+3)$
The roots are at -2 and -3 since
$f(-2) = 0$ and $f(-3) = 0$.

Are there any other x-values that produce a y-value of 0?
→ NO!



If *f(a)* = 0, then *a* must be a root of *f(x)*.
The corresponding factor is (x – a).

If you can find a value of *a* such that *f(a)* = 0, you will have found a factor of *f(x)*.

Ex. Factor $f(x) = x^3 + 3x^2 - 18x - 40$
If x = 1
$f(1) = 1^3 + 3(1)^2 - 18(1) - 40$
$= -54$
What does this represent?
$(x^3 + 3x^2 - 18x - 40) \div (x-1) = x^2 + 4x - 14$ *remainder* -54
∴ *f*(1) gives you the remainder when you divide by (x – 1)

If x =- 1
$f(-1) = (-1)^3 + 3(-1)^2 - 18(-1) - 40$
$= -20$
∴ *f*(-1) gives you the remainder when you divide by (x + 1)

If $x = 2$

$$f(2) = 2^3 + 3(2)^2 - 18(2) - 40$$
$$= -56$$

$\therefore$ $f(2)$ gives you the remainder when you divide by $(x - 2)$

If $x = -2$

$$f(-2) = (-2)^3 + 3(-2)^2 - 18(-2) - 40$$
$$= 0$$

$\therefore$ $f(-2)$ gives you the remainder when you divide by $(x + 2)$

Therefore $x = -2$ is a root of this equation and $(x + 2)$ is a factor.

$$f(x) = x^3 + 3x^2 - 18x - 40$$
$$= (x + 2)(???)$$

Returning to a numerical example:

If we know that 7 is a factor of 119, how do we find the other factor?

→ DIVIDE!

$119 \div 7 = 17$

$\therefore 119 = 7 \cdot 17$

The same applies to polynomials – we must divide *f(x)* by the factor $(x + 2)$.

Do you remember long division?!?

When we divide:

QUOTIENT

DIVISOR | DIVIDEND

REMAINDER

$\therefore$ Dividend = Quotient × Divisor + Remainder

Ex. $(2x^2 - 3x - 1) \div (x + 2)$

Ex. $(x^3 - 2x^2 + 2x - 15) \div (x - 3)$

Ex. $(-19x^2 + 6x^3 + 18x - 20) \div (2x - 5)$

HW: Pg. 168 #2, 3, 4

MHF4U – Unit 2: Polynomial Functions (Chapter 3 in book)

Lesson 6: Remainder & Factor Theorems – Part II

Recall:
We need to be able to factor higher order functions to find their roots.
To factor:

- Find a value of *a* such that *f(a)* = 0. The corresponding factor is (*x* – *a*).
- Divide *f(x)* by (*x* – *a*).
- Factor the quotient, if possible.

Synthetic division is a faster way of dividing a polynomial by a binomial in the form *ax* + *b*.

Ex. $\left(x^3 - 2x^2 + 2x - 15\right) \div (x - 3)$

$\Rightarrow$ Set the divisor equal to zero and solve (i.e. find the root)

$$x - 3 = 0$$
$$x = 3$$

$\Rightarrow$

3	1	-2	2	-15	$\leftarrow$ coefficients of dividend in descending order
		+	+	+	
		3	3	15	
	1	1	5	0	

$\therefore$ quotient is $x^2 + x + 5$, remainder is 0

Ex. $\left(-19x^2 + 6x^3 + 18x - 20\right) \div (2x - 5)$

$\Rightarrow$ Rewrite the dividend in standard form
$6x^3 - 19x^2 + 18x - 20$

$\Rightarrow$ Set the divisor equal to zero and solve

$$2x - 5 = 0$$
$$x = \frac{5}{2}$$

$\Rightarrow$ Divide then take each term of the quotient and divide it by the leading coefficient of the divisor. DO NOT DIVIDE THE REMAINDER.

$\frac{5}{2}$	6	-19	18	-20
		15	-10	20
	6	-4	8	0
	3	-2	4	

Divide by 2

$\therefore$ quotient is $3x^2 - 2x + 4$, remainder is 0

If the remainder is 0, the divisor and the quotient are both factors of the dividend.

If there is a term "missing", place a zero in its place.

The Remainder Theorem

When a polynomial P(x) is divided by $x - b$, the remainder is P(b).

When a polynomial P(x) is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$.

Ex. Determine the remainder when $2x^2 + 5x - 4$ is divided by $2x - 1$, without dividing.

Let $P(x) = 2x^2 + 5x - 4$

$$\begin{aligned}
P\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) - 4 \\
&= 2\left(\frac{1}{4}\right) + \frac{5}{2} - 4 \\
&= 3 - 4 \\
&= -1
\end{aligned}$$

$\therefore$ the remainder is -1

The Factor Theorem

A polynomial P(x) has the factor ($x - b$) if and only if P(b) = 0.

A polynomial P(x) has the factor ($ax - b$) if and only if $P\left(\frac{b}{a}\right)$=0.

To factor a polynomial of degree 3 or higher:

- ✓ Find a factor of the form ($x - b$): substitute values of b in $P(x)$ until you get a result of 0. Start with ± 1 then try factors of the constant term.
- ✓ Divide $P(x)$ by ($x - b$).
- ✓ Factor the result.

Ex. Factor $f(x) = x^3 + 2x^2 - 5x - 6$

1. f(1) = -8

   f(-1) = 0 $\therefore$ (x + 1) is a factor
2.

-1	1	2	-5	6
		-1	-1	6
	1	1	-6	0

   $\therefore f(x) = (x + 1)(x^2 + x - 6)$
3. Factoring we get:

   $f(x) = (x + 1)(x - 2)(x + 3)$

Ex. If (x – 2) is a factor of $f(x) = x^3 - 7x + k$, determine the value of k.

Since (x – 2) is a factor, $\therefore f(2) = 0$

$$\begin{aligned}
f(2) &= 2^3 - 7(2) + k \\
0 &= 8 - 14 + k \\
k &= 6
\end{aligned}$$

If none of the factors of the constant term give a remainder of 0, then the factor must be of the form *(ax – b)*.

Substitute values of the form $\frac{b}{a}$ where b are factors of the constant term and a are factors of the leading coefficient.

Ex. Determine a factor of $P(x) = 30x^3 + 17x^2 - 3x - 2$

$\Rightarrow$ Start by trying factors of the constant term:

$P(1) = 42$
$P(-1) = -12$
$P(2) = 300$
$P(-2) = -168$

$\Rightarrow$ Try factors of 2 over factors of 30

$\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{5}, \pm\frac{1}{6}, \pm\frac{1}{10}, \pm\frac{1}{15}, \pm\frac{1}{30}, \pm\frac{2}{3}, \pm\frac{2}{5}, \pm\frac{2}{15}$

$P\left(\frac{1}{2}\right) = 4.5$

$P\left(\frac{-1}{2}\right) = 0$

$\therefore$ there is a root at $x = \frac{-1}{2}$ and the factor is $(2x + 1)$.

Be sure to always write factors in the form $(ax + b)$ NOT $\left(x - \frac{b}{a}\right)$.

HW: Pg. 169 #6abd, 10be, 11, 12b, 13

T: Pg. 170 #14, 16

Pg. 177 #4b, 5b, 7ac, 9, 10, 12, 13, 14

MHF4U – Unit 2: Polynomial Functions (Chapter 3 in book)
Lesson 7: Sum and Difference of Cubes

Factoring the sum of cubes:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Factoring the difference of cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

HW: Pg. 182 #3, 5bd, 6 + unit 2 summary sheet

Lesson 1: Solving Polynomial Equations

Recall:

- Solve an equation means finding the real roots of the equation.
- There exist several methods of solving a quadratic equation: graphing, factoring, completing the square and using the quadratic formula.

Polynomial equations of degree 3 or higher can be solved by graphing (with technology) or factoring.

Ex. Solve $3x^3+8x^2=-3x+2$

1. Rewrite the equation so that it is equal to 0. You cannot solve an equation of this type without first setting it equal to 0!
   $3x^3+8x^2+3x-2=0$

2. Name the polynomial expression then find a factor.
   Let $P(x)=3x^3+8x^2+3x-2$
   $P(1) = 12$
   $P(-1) = 0 \quad \therefore (x+1)$ is a factor

3. Divide by the factor then factor the result.

$$\begin{array}{r|rrrr} -1 & 3 & 8 & 3 & -2 \\ & & -3 & -5 & 2 \\ \hline & 3 & 5 & -2 & 0 \end{array}$$

$\therefore\ 3x^2+5x-2$ is another factor

$$\therefore \begin{aligned} P(x) &= (x+1)\left(3x^2+5x-2\right) \\ &= (x+1)(x+2)(3x-1) \end{aligned}$$

4. Rewrite the equation and find the roots.
   $(x+1)(x+2)(3x-1)=0$

$$\therefore \begin{aligned} x+1&=0 \\ x&=-1 \end{aligned} \quad \text{or} \quad \begin{aligned} x+2&=0 \\ x&=-2 \end{aligned} \quad \text{or} \quad \begin{aligned} 3x-1&=0 \\ x&=\frac{1}{3} \end{aligned}$$

$\therefore$ The roots are -2, -1 and $\frac{1}{3}$.

Note: the roots can be checked by substitution.

Ex. Determine the exact roots of $x^3-4x^2+2x+3=0$.

Let $P(x)=x^3-4x^2+2x+3$
$P(1) = 2$
$P(-1) = -4$
$P(3) = 0 \quad \therefore (x-3)$ is a factor

3	1	-4	2	3	
		3	-3	-3	
	1	-1	-1	0	$\therefore\ x^2 - x - 1$ is another factor

$\therefore P(x) = (x-3)(x^2 - x - 1)$

$x^2 - x - 1$ cannot be factored

$\therefore$ we need to solve $(x-3)(x^2 - x - 1) = 0$

$$\therefore \quad \begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

$\therefore$ The exact roots are 3, $\dfrac{1-\sqrt{5}}{2}$, $\dfrac{1+\sqrt{5}}{2}$.

Ex. A box is in the shape of a rectangular prism. The length is 12 cm greater than the width and the ends are square. The volume of the box is 135 cm^3. What are the dimensions of the box?

Let x be the width $\therefore$ the height is also x and the length is x + 12.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= (x+12)(x)(x) \\ 135 &= x^3 + 12x^2 \\ x^3 + 12x^2 - 135 &= 0 \end{aligned}$$

Let
$P(x) = x^3 + 12x^2 - 135$
$P(1) = -122$
$P(-1) = -124$
$P(3) = 0 \qquad \therefore (x-3)$ *is a factor*

3	1	12	0	-135	
		3	45	135	
	1	15	45	0	$\therefore\ x^2 + 15x + 45$ is another factor

$\therefore$ we need to solve $(x-3)(x^2 + 15x + 45) = 0$

$$x = 3 \qquad \text{or} \qquad \begin{aligned} x &= \frac{-15 + \sqrt{225 - 4(1)(45)}}{2} \\ &= \frac{-15 + \sqrt{45}}{2} \\ &\doteq -4.14 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \frac{-15 - \sqrt{225 - 4(1)(45)}}{2} \\ &= \frac{-15\sqrt{45}}{2} \\ &\doteq -10.85 \end{aligned}$$

Only the first root is admissible as lengths cannot be negative $\therefore$ the dimensions of the box are 15 cm by 3 cm by 3cm.

HW: Pg. 204 #6, 7d, 13

Lesson 2: Solving Linear Inequalities

Recall:

Relational operators are used to express inequalities:

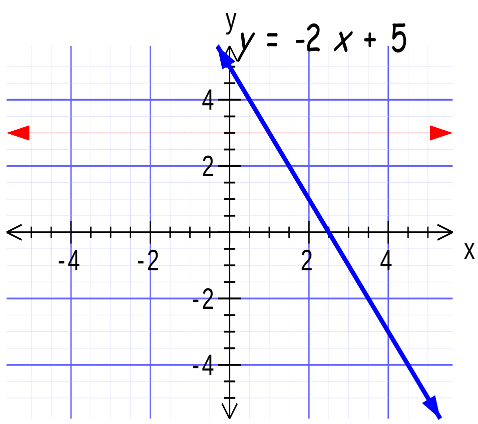
- $>$ greater than
- $\geq$ greater than or equal to
- $<$ less than
- $\leq$ less than or equal to

Interval notation:

- Use square brackets for values that are included
- Use round brackets for values that are not included

  Ex. $-3 < x \leq 8 \quad \rightarrow \quad (-3, 8]$

Ex. Solve $-2x + 5 < 3$



→ We want to know where the graph of $y = -2x + 5$ has a value less than 3.

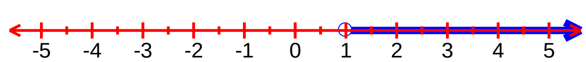
→ Solution is $x > 1$

Algebraically:

$$-2x + 5 < 3$$
$$-2x < -2$$
$$\frac{-2x}{-2} > \frac{-2}{-2}$$
$$x > 1$$

Note: When you multiply or divide by a negative, you must change the direction of the inequality.

The solution can be expressed on a number line:

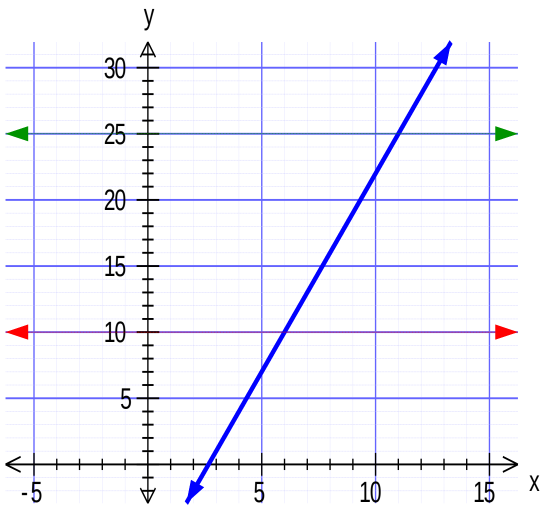


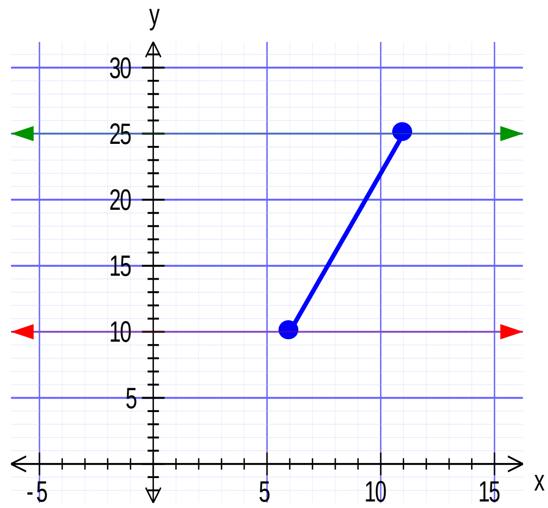
Ex. Solve $10 \leq 3(2x - 5) - (3x - 7) \leq 25$

This is a **double inequality**. The solution must satisfy

$10 \leq 3(2x - 5) - (3x - 7)$ **AND** $3(2x - 5) - (3x - 7) \leq 25$

The function defined by $y = 3(2x-5)-(3x-7)$ must be greater than or equal to 10 and less than or equal to 25:

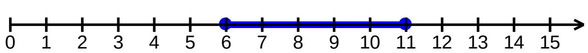




You can solve these two inequalities algebraically at the same time:

$$
\begin{aligned}
10 &\le 3(2x-5)-(3x-7) \le 25 \\
10 &\le 6x-15-3x+7 \le 25 && \leftarrow \text{Expand and simplify} \\
10 &\le 3x-8 \le 25 \\
10+8 &\le 3x-8+8 \le 25+8 && \leftarrow \text{Add 8 to all three parts} \\
18 &\le 3x \le 33 && \leftarrow \text{Divide all three parts by 3} \\
6 &\le x \le 11
\end{aligned}
$$

Expressed on a number line:



HW: Pg. 213 #4de, 5bef, 7ef, 8a, 9, 10, 11

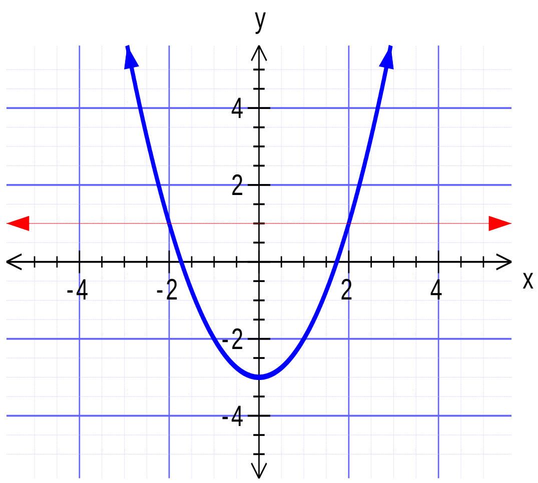
Lesson 3: Solving Polynomial Inequalities

Recall:

- To solve a polynomial equation, set it equal to zero and solve.
- Solving inequalities yields an interval as the solution set.

Ex. Solve $x^2 - 3 < 1$

$\rightarrow$ We want to find the interval over which the ordinate (y-value) of the parabola defined by $f(x) = x^2 - 3$ is less than 1.



Graphically we get -2 < x < 2 as a solution.

Algebraically:

$x^2 - 3 < 1$ $\qquad\leftarrow$ this is equivalent to finding the interval where

$x^2 - 4 < 0$ $\qquad$ $y = x^2 - 4$ is below the x-axis

The endpoints of the solution set are the roots of the new function $y = x^2 - 4$.

$\rightarrow$ Find the roots:

$$x^2 - 4 = 0$$
$$(x+2)(x-2) = 0$$

$\therefore$ $x = -2$ or $x = 2$

$\rightarrow$ Use a table to look at the sign of the function on either side of each root:

-2 $\qquad$ 2

	Intervals →	x < -2	-2 < x < 2	x > 2
Factors	x + 2	-	+	+
	x - 2	-	-	+
Polynomial →	x^2 - 4	+	-	+

$\therefore$ $x^2 - 4 < 0$ when -2 < x < 2

$\therefore$ $x^2 - 3 < 1$ when -2 < x < 2

Ex. Solve the following inequality algebraically.

$$-x^4 + x^3 + 17x^2 - 21x - 30 > 6$$
$$-x^4 + x^3 + 17x^2 - 21x - 36 > 0$$

Let $P(x) = -x^4 + x^3 + 17x^2 - 21x - 36$

$P(1) = -40$

$P(-1) = 0 \quad \therefore (x + 1)$ is a factor

$$\begin{array}{r|rrrrr} -1 & -1 & 1 & 17 & -21 & -36 \\ & & 1 & -2 & -15 & 36 \\ \hline & -1 & 2 & 15 & -36 & 0 \end{array}$$

$\therefore \ -x^3 + 2x^2 + 15x - 36$ is a factor

Let $Q(x) = -x^3 + 2x^2 + 15x - 36$

$Q(1) = -20$

$Q(-1) = -48$

$Q(2) = -6$

$Q(-2) = -50$

$Q(3) = 0 \quad \therefore (x - 3)$ is a factor

$$\begin{array}{r|rrrr} 3 & -1 & 2 & 15 & -36 \\ & & -3 & -3 & 36 \\ \hline & -1 & -1 & 12 & 0 \end{array}$$

$\therefore -x^2 - x + 12$ is a factor

$$\begin{aligned} P(x) &= (x+1)(x-3)\left(-x^2 - x + 12\right) \\ &= -(x+1)(x-3)\left(x^2 + x - 12\right) \\ &= -(x+1)(x-3)(x+4)(x-3) \\ &= -(x+1)(x-3)^2(x+4) \end{aligned}$$

$\therefore$ there are roots at -4 and -1 and a double-root at 3

We need to solve:

$$-(x+1)(x-3)^2(x+4) > 0$$

OR

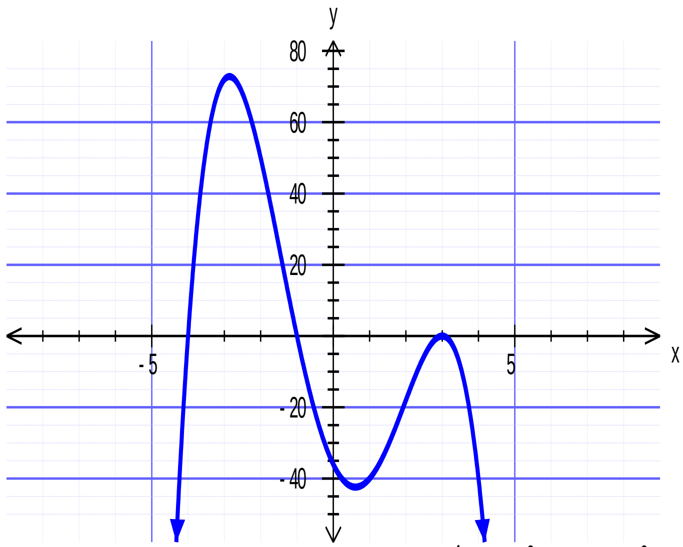
$$(x+1)(x-3)^2(x+4) < 0$$

	-4	-1	3	
	x < -4	-4 < x < -1	-1 < x < 3	x > 3
x + 1	-	-	+	+
$(x-3)^2$	+	+	+	+
x + 4	-	+	+	+
product	+	-	+	+

← always positive

$\therefore -x^4 + x^3 + 17x^2 - 21x - 30 > 6$ when $-4 < x < -1$

Sketching confirms the result:



$y = -x^4 + x^3 + 17x^2 - 21x - 36$

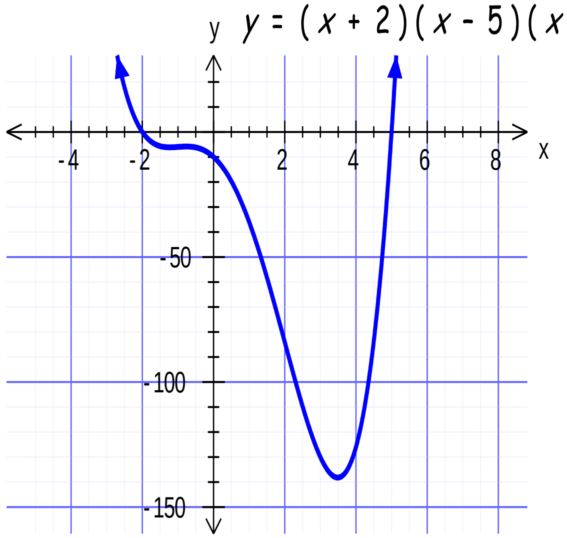
Ex. Solve $(x+2)(x-5)(x^2+x+1)<0$

$\rightarrow$ If we try to solve $x^2+x+1=0$ we get a negative discriminant $\therefore$ the only real roots are -2 and 5.

	-2	5	
	x < -2	-2 < x < 5	x > 5
x + 2	-	+	+
x - 5	-	-	+
$x^2 + x + 1$	+	+	+
product	+	-	+

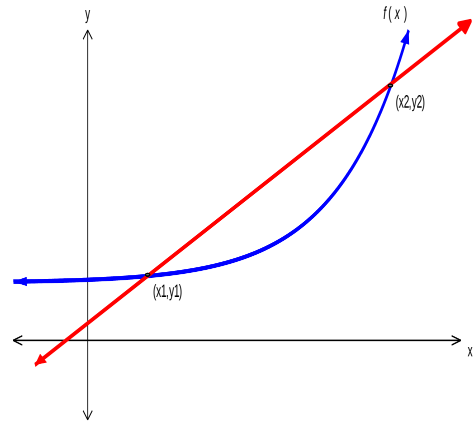
$\therefore (x+2)(x-5)(x^2+x+1)<0$ when $-2 < x < 5$

$y = (x+2)(x-5)(x^2+x+1)$



HW: Pg. 226 #6cd, 7cf (no graph), 8, 10

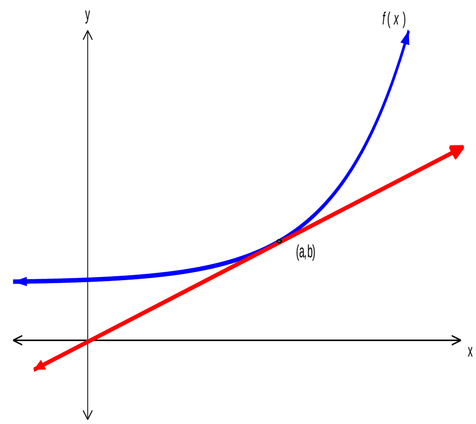
Lesson 4: Rates of Change



The **average rate of change** for any function *f(x)* over the interval $x_1 \le x \le x_2$ is equivalent to the slope of the secant passing through the points (x_1, y_1) and (x_2, y_2).

Average Rate of Change:

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



The **instantaneous rate of change** at a point is equivalent to the slope of the tangent at the point.

Since we cannot find the slope of a line through only one point, estimate the instantaneous rate of change for any function *f(x)* use a series of secants whose slope **approaches** that of the tangent. Look at the slope of the secant for preceding intervals and following intervals (i.e. to the left of $x=a$ and to the right of $x=a$). For example, to estimate the instantaneous rate of change at $x = a$.

The smaller we make Δx, the closer the secant comes to the tangent. They would be the same if $\Delta x = 0$, but we can only **approach** 0.

Δt	$v_{avg} = \frac{\Delta d}{\Delta t}$
0.1	11.64
0.01	11.39
0.001	11.36
↓	↓
0	v_{inst}

$$m_{tan} = \lim_{\Delta t \to 0} m_{sec}$$

$$m_{tan} = \lim_{\Delta t \to 0} \frac{\Delta d}{\Delta t}$$

$$\downarrow$$

$$v_{inst} = \lim_{\Delta t \to 0} v_{avg}$$

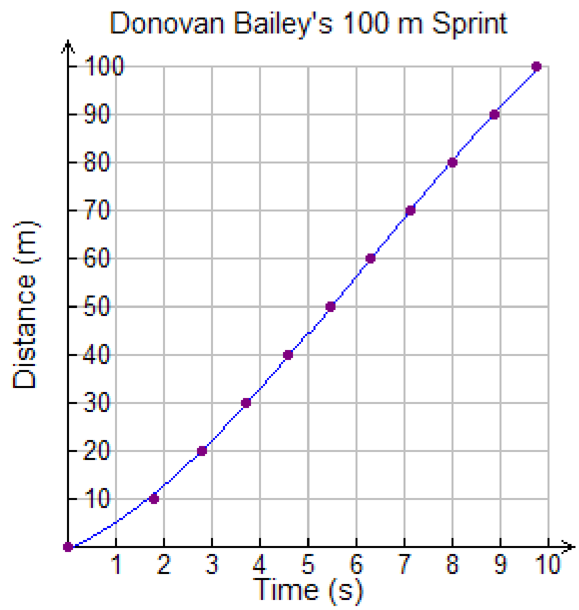
HW: Pg. 95 #1, 3, 6; Pg. 103 #1, 3, 5, 8

During the 1997 World Championships in Athens, Greece, Maurice Greene and Donovan Bailey ran a 100 m race.
(http://hypertextbook.com/facts/2000/KatarzynaJanuszkiewicz.shtml)

Part A

The graph and table below show Donovan Bailey's performance during this 100 m race.

Donovan Bailey's Performance	
Time (s)	**Distance (m)**
0	0
1.78	10
2.81	20
3.72	30
4.59	40
5.44	50
6.29	60
7.14	70
8.00	80
8.87	90
9.77	100



1. a) Calculate Donovan Bailey's average velocity for this 100 m sprint.

$$\text{Average Velocity} = \frac{\textit{change in distance}}{\textit{change in time}} = \frac{\Delta d}{\Delta t}$$

b) Draw a line from (0,0) to (9.77,100) on the graph above.

A line passing through at least two different points on a curve is called a **secant**.

c) Explain the relationship between your answer to a) and the slope of the secant.

2. a) Draw the secants from (0,0) to (5.44,50) and from (5.44,50) to (9.77,100).

b) Calculate the average velocities represented by the two secants drawn in a).

i) ii)

c) Compare Bailey's performance during the first and the second half of the race.

3. Describe the relationship between average velocity and the slope of the corresponding secant.

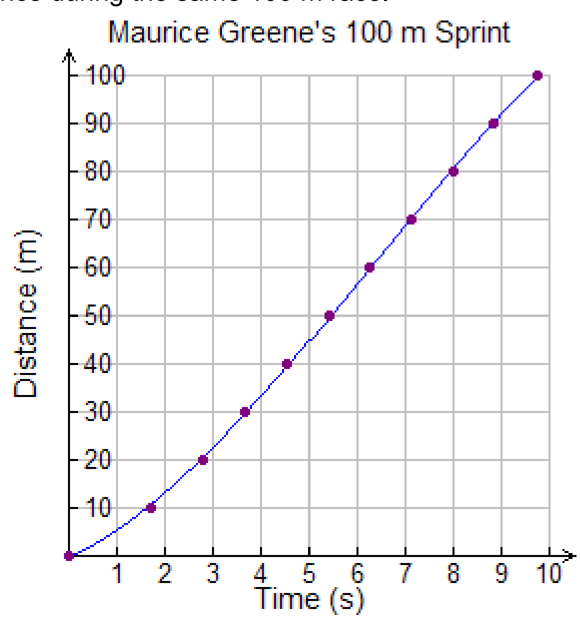
4. Calculate Bailey's average velocity for each 10 m interval of this 100 m race. Record your answers in the table below.

Interval (m)	Distance Travelled Δd (m)	Time Elapsed Δt (s)	Average Velocity (m/s)
0 to 10			
10 to 20			
20 to 30			
30 to 40			
40 to 50			
50 to 60			
60 to 70			
70 to 80			
80 to 90			
90 to 100			

Part B

The graph and table show Maurice Greene's performance during the same 100 m race.

Maurice Greene's Performance	
Time (s)	**Distance (m)**
0	0
1.71	10
2.75	20
3.67	30
4.55	40
5.42	50
6.27	60
7.12	70
7.98	80
8.85	90
9.73	100



Calculate Greene's average velocity for each 10 m interval of this 100 m race. Record your answers in the table below.

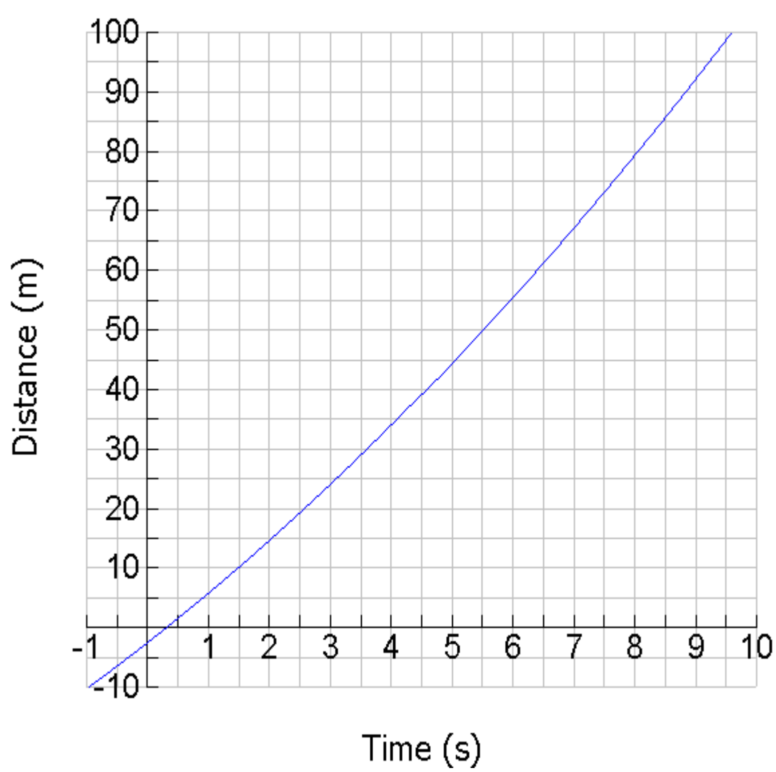
Interval (m)	Distance Travelled Δd (m)	Time Elapsed Δt (s)	Average Velocity (m/s)
0 to 10			
10 to 20			
20 to 30			
30 to 40			
40 to 50			
50 to 60			
60 to 70			
70 to 80			
80 to 90			
90 to 100			

Part C

Using your calculations from Parts A and B, describe this 100 m race run by Donovan Bailey and Maurice Greene. Include who was fastest and who was leading at various points during the race.

In 1997, Donavan Bailey ran the 100 m sprint in 9.77 seconds. The table below describes his run. One model that describes this run is a quadratic model with an equation of:
d(t) = 0.28t2 + 8.0t – 2.54.

Time (s)	Distance (m)
0	0
1.78	10
2.81	20
3.72	30
4.59	40
5.44	50
6.29	60
7.14	70
8.00	80
8.87	90
9.77	100



1. a) Estimate Donavan Bailey's **instantaneous** velocity at *t* = 6 s.

b) Explain why you think your answer to a) is a good approximation.

c) Plot a point on the curve at 6 seconds. Draw a line that passes through this point but does not pass through the curve again. This line is called a **tangent** to the curve.

2. Use the algebraic model $d(t) = 0.28t^2 + 8.0t - 2.54$ to approximate the instantaneous velocity of Donavan Bailey at t = 6 s, by completing the charts below, using a graphing calculator.

Interval	Interval Endpoints	Δd	Δt	Average Velocity: $\frac{\Delta d}{\Delta t}$
$6 \le t \le 7$				
$6 \le t \le 6.1$				
$6 \le t \le 6.01$				
$6 \le t \le 6.001$				

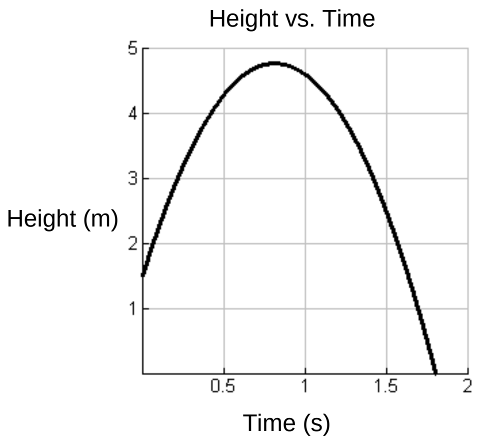
Interval	Interval Endpoints	Δd	Δt	Average Velocity: $\frac{\Delta d}{\Delta t}$
$5 \le t \le 6$				
$5.9 \le t \le 6$				
$5.99 \le t \le 6$				
$5.999 \le t \le 6$				

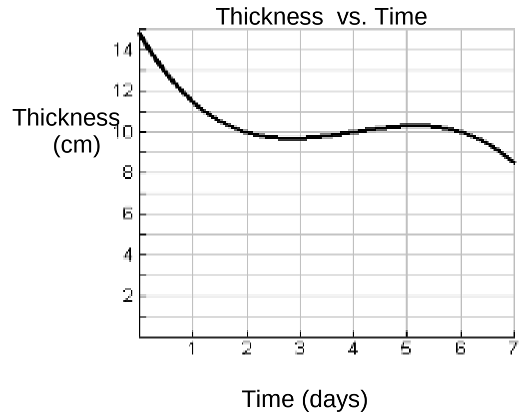
i) Use the calculations from the charts to estimate the instantaneous velocity at t = 6 s.

j) If you could draw the secants that correspond with the average velocities calculated above, how would they compare to the tangent drawn in 1(c)?

1. Using the graphs below, estimate the instantaneous rates of change (m_T) for each of the graphs at the given points.

A.





t	0	0.25	0.5	0.8	1	1.5	1.8	*t*	0	1	2	3	4	5	6	7
m_T								m_T								

2. State the domain and the range of the two functions.

Graph A
Domain:
Range:

Graph B
Domain:
Range:

3 a) Describe the graphical features (e.g., local maximum/minimum point, interval of increase/decrease), m_T values (e.g., +, -, 0), and, where appropriate, the trend of the slope of the tangent (e.g., changing from positive to zero to negative) as time increases.

Interval	Graphical feature	m_T values	m_T trend (if appropriate)
Graph A: Domain 0-0.8			
Graph A: at 0.8			
Graph A: Domain 0.8–1.75			
Graph B: Domain 0–3			
Graph B: at 3			
Graph B: Domain 3–5			
Graph B: at 5			
Graph B: Domain 5–7			

b) Describe the context where the slope of the tangent is zero. What does it mean?

c) What are the similarities and differences between the slopes of the tangents?

4. The slope of the secant line can be a good estimate of the slope of the tangent. Rod thought that using an interval of 1 second to determine the slope of the secant line in graph A is good enough to estimate the slope of the tangent. Do you agree or disagree? Justify your reasoning.

5. For Graph A, state the slope of the tangent at 0.5 and 1 second. At which point is the shot put going faster? Explain.

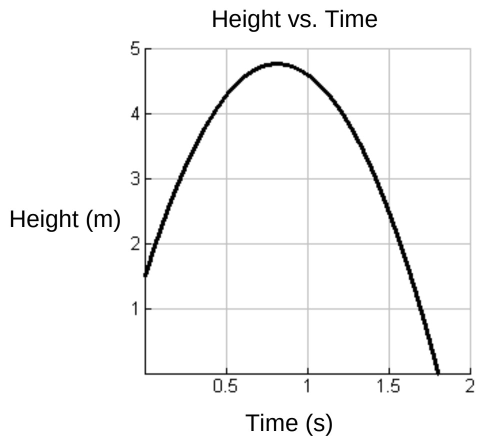
6. Draw three or four curves that have a secant slope of 2 and that pass through the same two points. What inference can be made from this?

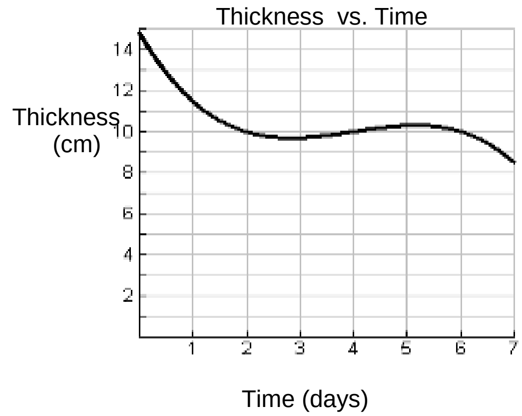
7. Anne says that a tangent crosses a curve in one and only one point. Do you agree or disagree? Use Graph B to justify your position.

MHF 4U   Tangent Slopes and Graph Characteristics

1. Using the graphs below, estimate the instantaneous rates of change (m_T) for each of the graphs at the given points.

A.





t	0	0.25	0.5	0.8	1	1.5	1.8	*t*	0	1	2	3	4	5	6	7
m_T								m_T								

2. State the domain and the range of the two functions.

Graph A
Domain:
Range:

Graph B
Domain:
Range:

3 a) Describe the graphical features (e.g., local maximum/minimum point, interval of increase/decrease), m_T values (e.g., +, -, 0), and, where appropriate, the trend of the slope of the tangent (e.g., changing from positive to zero to negative) as time increases.

Interval	Graphical feature	m_T values	m_T trend (if appropriate)
Graph A: Domain 0-0.8			
Graph A: at 0.8			
Graph A: Domain 0.8–1.75			
Graph B: Domain 0–3			
Graph B: at 3			
Graph B: Domain 3–5			
Graph B: at 5			
Graph B: Domain 5–7			

b) Describe the context where the slope of the tangent is zero. What does it mean?

c) What are the similarities and differences between the slopes of the tangents?

MHF 4U Tangent Slopes and Graph Characteristics (continued)

4. The slope of the secant line can be a good estimate of the slope of the tangent. Rod thought that using an interval of 1 second to determine the slope of the secant line in graph A is good enough to estimate the slope of the tangent. Do you agree or disagree? Justify your reasoning.

5. For Graph A, state the slope of the tangent at 0.5 and 1 second. At which point is the shot put going faster? Explain.

6. Draw three or four curves that have a secant slope of 2 and that pass through the same two points. What inference can be made from this?

7. Anne says that a tangent crosses a curve in one and only one point. Do you agree or disagree? Use Graph B to justify your position.

MFH4U – Unit 3

Lesson 5: Estimating Instantaneous Rates of Change

Recall:

- ✓ To find the equation of a line you need the slope, m, and a point (x_1,y_1) on the line: $y - y_1 = m(x - x_1)$
- ✓ The slope of a secant line represents the average rate of change between two points on the graph of a relation.
- ✓ The slope of the tangent line represents the instantaneous rate of change at a specific point on the graph of a relation.
- ✓ $m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ where the limit represents the value that the ratio approaches as Δx gets closer and closer to 0.

Ex. Determine the slope of the tangent to $y = 2x^2$ at P(1,2).

→ We need another point on the curve: $Q(x, 2x^2)$

→ Create a series of secant lines through P and Q…

	Q	P	Δy	Δx	m_{PQ}
From the right	(2,)	(1,2)			
	(1.5,)				
	(1.1,)				
	(1.01,)				
From the left	(0,0)				
	(0.5,)				
	(0.9,)				
	(0.99,)				

As Q approaches P:

→ From the right, the slope of PQ is getting smaller and appears to be approaching 4.

→ From the left, the slope of PQ is getting larger and appears to be approaching 4.

∴ the slope of the tangent to P will be exactly 4 when $\Delta x = 0$.

$$m_{tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= 4$$

Ex. Determine the equation of the tangent to $y = 2x^3 + x - 3$ at the point where $x = 2$.

→ Find the point of tangency:

when $x = 2$, $y = 15$ ∴ (2,15)

→ Find an expression for the slope of the secant:

$$\frac{\Delta y}{\Delta x} = \frac{f(x)-15}{x-2}$$
$$= \frac{(2x^3+x-3)-15}{x-2}$$
$$= \frac{2x^3+x-18}{x-2}$$
$$= 2x^2+4x+9$$

$$\therefore\ m_{tan} = \lim_{x\to 2}(2x^2+4x+9)$$

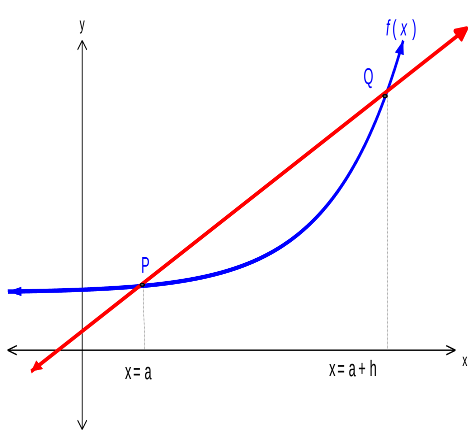
From the left		From the right	
x	**$2x^2$ + 4x + 9**	**x**	**$2x^2$ + 4x + 9**
1		3	
1.5		2.5	
1.9		2.1	
1.99		2.01	
1.999		2.001	

$\therefore$ m_{tan} = 25

→ Equation of tangent:

$$y - y_1 = m(x - x_1)$$
$$y - 15 = 25(x-2)$$
$$y = 25x - 35$$

We can generalize the expression for the secant:



The coordinates of the two points are:
P(a, f(a)) and Q(a+h, f(a+h))

$$\frac{\Delta y}{\Delta x} = \frac{y_Q - y_P}{x_Q - x_P}$$
$$= \frac{f(a+h)-f(a)}{a+h-a}$$
$$= \frac{f(a+h)-f(a)}{h}$$

We can use this expression for very small positive and negative values of *h*.

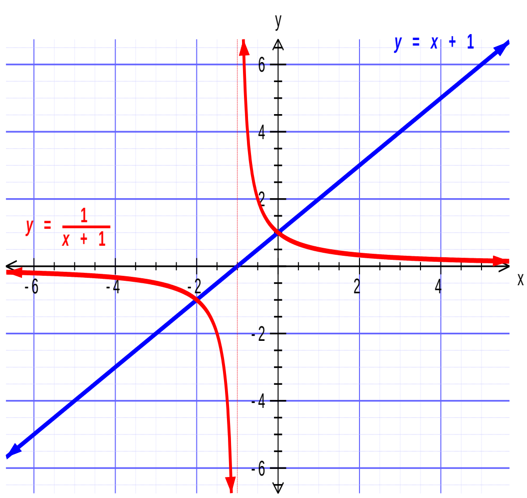
HW: Pg. 236 #5ad, 6ad, 7

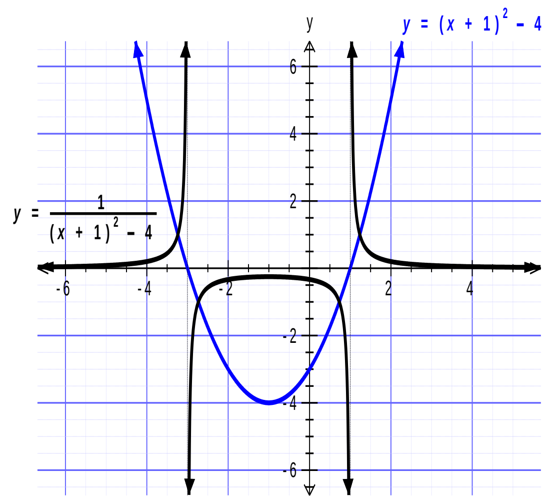
Lesson 1: Graphs of Reciprocal Functions

Do "INVESTIGATE the Math" on pages 248-249.

Summary:

- ✓ If a function has coordinates $\left(x, \frac{a}{b}\right)$, the reciprocal function has coordinates $\left(x, \frac{b}{a}\right)$.
- ✓ The graph of a reciprocal function has a vertical asymptote at each zero of the original function.
- ✓ If the original function is linear or quadratic, its reciprocal function will always have $y = 0$ as a horizontal asymptote.
- ✓ A reciprocal function has the same positive/negative intervals as the original function.
- ✓ Intervals of increase on the original function are intervals of decrease on the reciprocal function, and vice-versa.
- ✓ If the range of the original function includes 1 and/or -1, the reciprocal function will intersect the original function at a point(s) where the y-coordinate is 1 or -1.
- ✓ If the original function has a local minimum point, the reciprocal function will have a local maximum point at the same x-value, and vice versa.





HW: Pg. 254 #1, 2cd, 7a, 8f, 10

T: Pg. 257 #11, 13

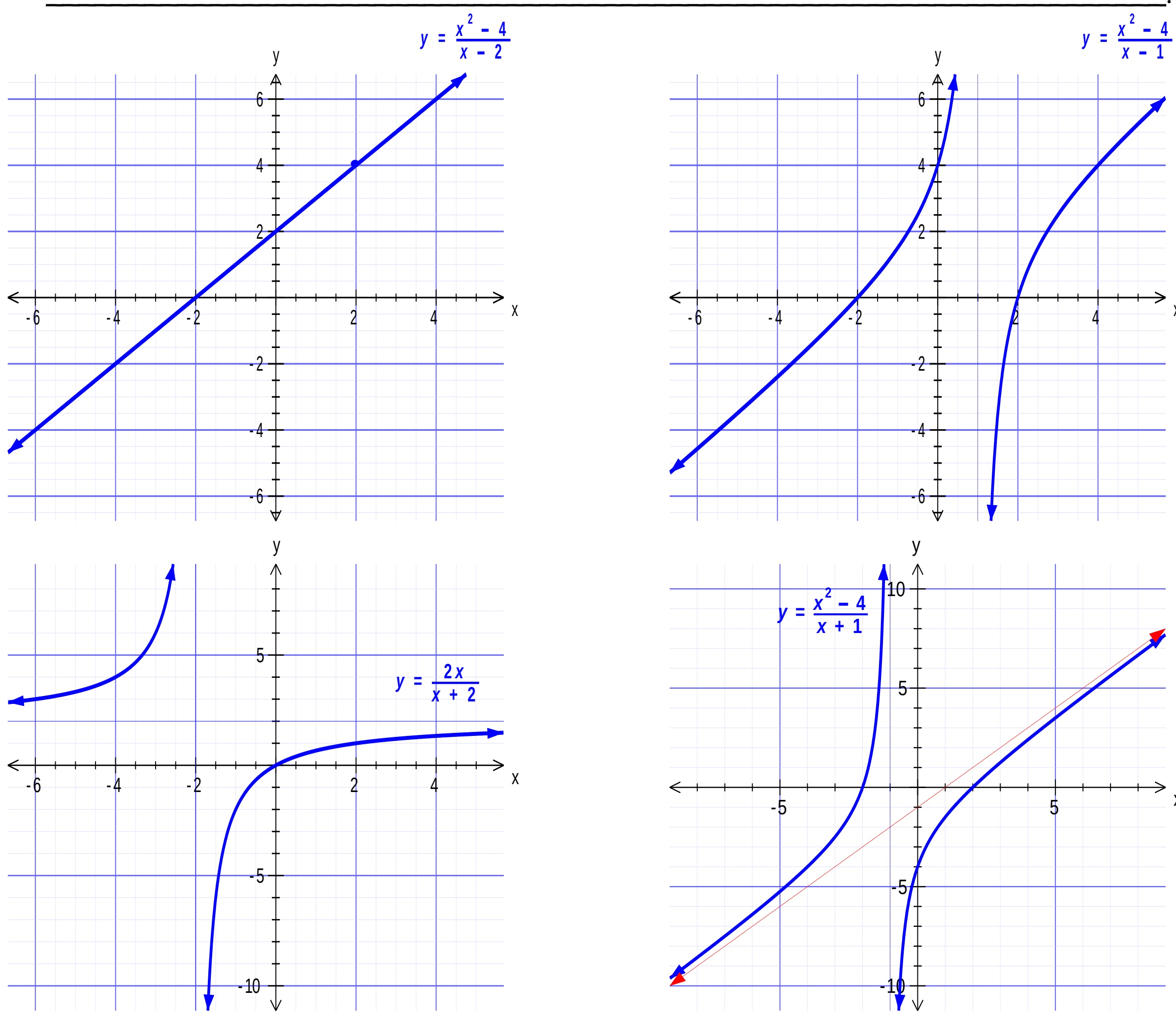
MFH4U – Unit 4: Rational Functions, Equations and Inequalities

Lesson 2: Exploring Quotients of Rational Functions

Do "EXPLORE the Math" on pages 258-259 (not C & D).

Summary:

- ✓ A **rational function** is a function that can be expressed as $f(x)=\dfrac{p(x)}{q(x)}$, where *p(x)* and *q(x)* are polynomial functions, $q(x)\neq 0$.
- ✓ A rational function, *f(x)*, has a hole at $x = a$ if ____.
- ✓ A rational function, *f(x)*, has a vertical asymptote $x = a$ if ____.
- ✓ A rational function, $f(x)=\dfrac{p(x)}{q(x)}$, has a horizontal asymptote only when ____.
- ✓ A rational function, $f(x)=\dfrac{p(x)}{q(x)}$, has an **oblique** (slanted) **asymptote** only when ____.



HW: Pg. 262 #1, 2bcdj, 3

Lesson 3: Graphs of Rational Functions of the Form $f(x)=\dfrac{ax+b}{cx+d}$

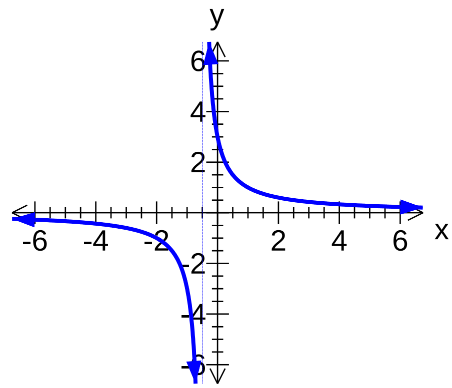
Finding horizontal asymptotes

Horizontal asymptotes affect the end behaviour of graphs so we need to look at the value of the function as $x \to \infty$ and $x \to -\infty$.

Ex. $f(x)=\dfrac{3}{2x-1}$

$x \to \infty$		$x \to -\infty$.	
x	***f(x)***	***x***	***f(x)***
10		-10	
100		-100	
1000		-1000	
10000		-10000	
100000		-100000	

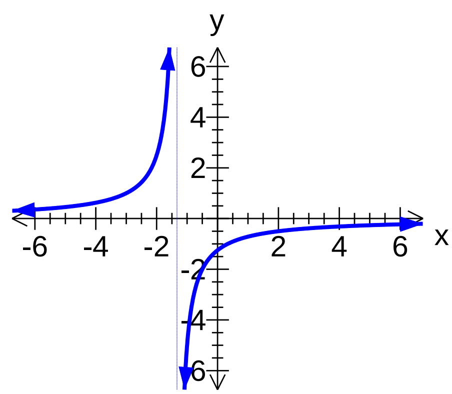
$\therefore$ There is a horizontal asymptote at $y = 0$



Ex. $f(x)=\dfrac{-5}{3x+4}$

$x \to \infty$		$x \to -\infty$.	
x	***f(x)***	***x***	***f(x)***
10		-10	
100		-100	
1000		-1000	
10000		-10000	
100000		-100000	

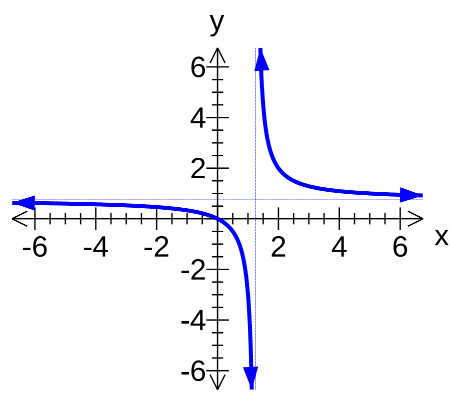
$\therefore$ There is a horizontal asymptote at $y = 0$



Ex. $f(x)=\dfrac{3x}{4x-5}$

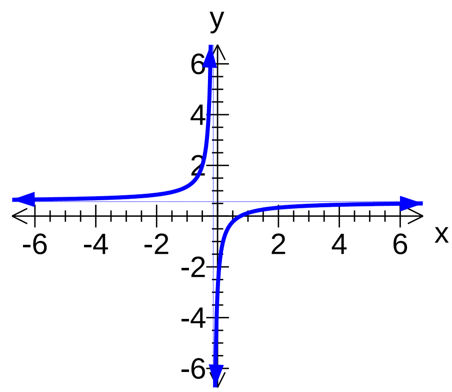
$x \to \infty$		$x \to -\infty$.	
x	***f(x)***	***x***	***f(x)***
10		-10	
100		-100	
1000		-1000	
10000		-10000	
100000		-100000	

$\therefore$ There is a horizontal asymptote at $y = \dfrac{3}{4}$



Ex. $f(x)=\dfrac{4x-3}{7x+1}$

$x \to \infty$		$x \to -\infty$.	
x	***f(x)***	***x***	***f(x)***
10		-10	
100		-100	
1000		-1000	
10000		-10000	
100000		-100000	



$\therefore$ There is a horizontal asymptote at $y = \frac{4}{7}$

Conclusions:

- Rational functions of the form $f(x) = \frac{b}{cx+d}$ have a horizontal asymptote defined by $y = 0$.

- Most rational functions of the form $f(x) = \frac{ax+b}{cx+d}$ have a horizontal asymptote defined by $y = \frac{a}{c}$.

Vertical Asymptotes

Vertical asymptotes (VA) occur where the function is undefined. Determine the equation of vertical asymptotes by setting the denominator equal to zero and solving.

Exception

If the numerator and denominator share a common factor, there will be a hole where the zero of the common factor occurs → this will affect both horizontal and vertical asymptotes so you must **always** check for common factors first!

Ex. $f(x) = \frac{3x-6}{x-2}$

$$f(x) = \frac{3(x-2)}{x-2}$$

$$f(x) = 3,\ x \neq 2$$

$\rightarrow$ no VA, no HA

Key Characteristics for Sketching

- Domain
- Intercepts
- Asymptotes
- Positive/negative intervals

Ex. Sketch $f(x) = \frac{4x-10}{2x+5}$

- Domain

  $\left\{ x \in \Re \mid x \neq -\frac{5}{2} \right\}$

- Intercepts

  When $x = 0$, $y = -2$ $\Rightarrow$ y-int at -2

$$\frac{4x-10}{2x+5} = 0$$

$$4x - 10 = 0$$

$$x = \frac{10}{4}$$

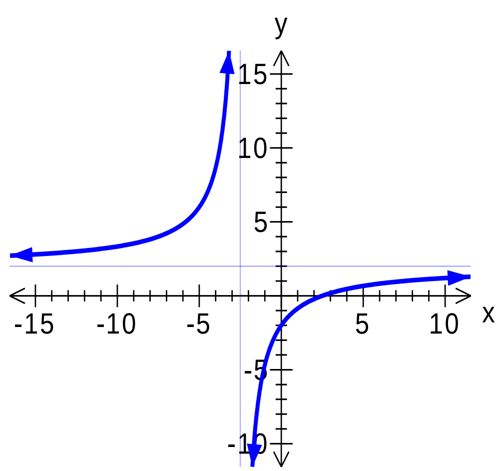
$$x = \frac{5}{2}$$

$\Rightarrow$ x-int at $\frac{5}{2}$

- Asymptotes
  $\rightarrow$ Function cannot be simplified…
  VA:
  $$2x + 5 = 0$$
  $$x = -\frac{5}{2}$$
  $$x = -2.5$$
  HA:
  $$y = \frac{4}{2}$$
  $$y = 2$$

- Positive/negative intervals
  The x-intercepts and vertical asymptotes are where the function may change signs.

	$x < -\frac{5}{2}$	$-\frac{5}{2} < x < \frac{5}{2}$	$x > \frac{5}{2}$
4x - 10	-	-	+
2x + 5	-	+	+
quotient	+	-	+



HW: Pg. 272 #1, 5bcd, 6b

T: Pg. 274 #13, 14

MFH4U – Unit 4: Rational Functions, Equations and Inequalities
Lesson 4: Solving Rational Equations

Recall:
Determine the LCM of the denominators for each pair of rational expressions:

a) $\frac{7}{6x^3}, \frac{x}{14x^4}$

b) $\frac{x}{x+2}, \frac{2}{x^2}$

c) $\frac{2}{x^2-1}, \frac{3}{x+1}$

Ex. When Ian and Janet work together they can paint the exterior of a home in 7 working days. When Janet works alone she takes 2 days more than Ian does when he works alone. How long does Ian take if he works alone?

Let d be the number of days Ian takes when he works alone.
$\therefore$ Janet takes (d + 2) days.

The fraction of house painted in 1 day:

- Ian alone: $\frac{1}{d}$
- Janet alone: $\frac{1}{d+2}$
- Ian & Janet together: $\frac{1}{7}$

$$\therefore \frac{1}{d}+\frac{1}{d+2}=\frac{1}{7}$$

To solve you must multiply each term by the LCM of the denominator. Here, LCM = $7d(d + 2)$

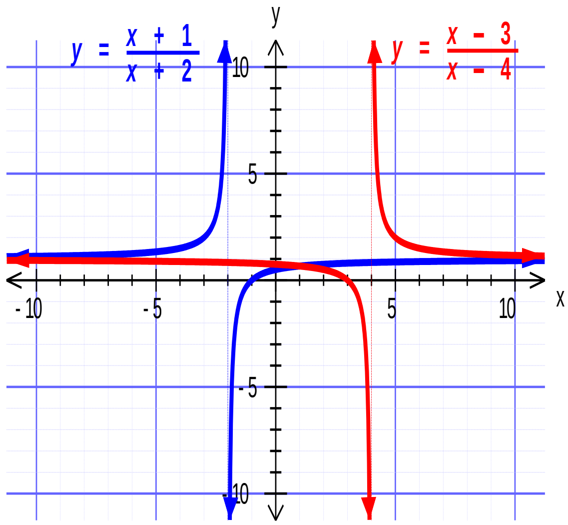
$$7d(d+2)\left[\frac{1}{d}\right]+7d(d+2)\left[\frac{1}{d+2}\right]=7d(d+2)\left[\frac{1}{7}\right]$$
$$7(d+2)+7d=d(d+2)$$
$$7d+14+7d=d^2+2d$$
$$d^2-12d-14=0$$

$$d \doteq -1.07 \quad or \quad d \doteq 13.07$$

Since d > 7, Ian takes about 13 days to paint the house alone.

Ex. Solve $\frac{x+1}{x+2}=\frac{x-3}{x-4}$

$$(x+2)(x-4)\left(\frac{x+1}{x+2}\right)=(x+2)(x-4)\left(\frac{x-3}{x-4}\right)$$
$$(x-4)(x+1)=(x+2)(x-3)$$
$$x^2-3x-4=x^2-x-6$$
$$-2x=-2$$
$$x=1$$



Ex. A group of friends planned a trip and chartered a bus for \$247.50. When they got four more friends to join the trip, the cost of the bus was \$6 less per person. In all, how many people made the trip?

Let x be the original number of people.

- Original price per person: $\frac{247.50}{x}$
- New price per person: $\frac{247.50}{x+4}$

$$\therefore \frac{247.50}{x} - 6 = \frac{247.50}{x+4}$$
$$(x)(x+4)\frac{247.50}{x} - 6(x)(x+4) = (x)(x+4)\frac{247.50}{x+4}$$
$$247.50(x+4) - 6x(x+4) = 247.50x$$
$$247.50x + 990 - 6x^2 - 24x = 247.50x$$
$$-6x^2 - 24x + 990 = 0$$
$$x^2 + 4x - 165 = 0$$
$$(x+15)(x-11) = 0$$

$\therefore x = -15$ or $x = 11$

Since $x > 0$, there were originally 11 people planning the trip, but 15 people made the trip.

HW: Pg. 285 #1, 5ace, 6be, 9

Day 2 (after lesson 5) : Pg. 286 #10, 11, 12, 13 + Pg. 297 #9, 11, 12

MHF4U – Unit 4

Lesson 5: Solving Rational Inequalities

Rational inequalities can be solved in a similar manner to polynomial inequalities.

Ex. Solve $x-1<\frac{12}{x}$

Since we do not know if x is positive or negative, we cannot multiply each term by x as doing so might change the direction of the inequality. Instead, move all the terms to one side so that the other side is 0 then find a common denominator so that the entire expression is written as one fraction.

$$x-1-\frac{12}{x}<0$$

$$\frac{x^2}{x}-\frac{x}{x}-\frac{12}{x}<0$$

$$\frac{x^2-x-12}{x}<0$$

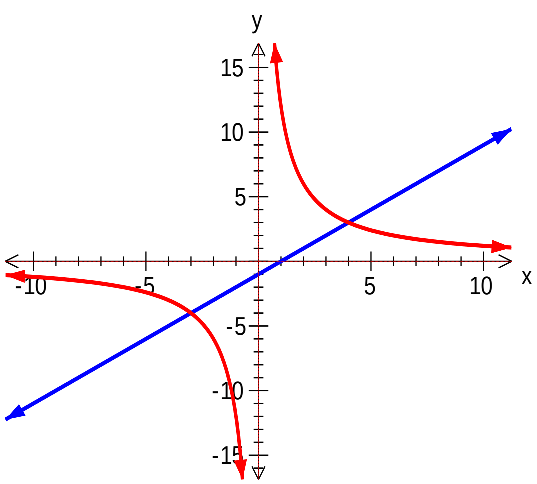
$$\frac{(x-4)(x+3)}{x}<0$$

Use a sign table to determine the intervals that are part of the solution set.

The roots of the numerator are 4 and -3. The root of the denominator is 0 → this represents a vertical asymptote.

	x < -3	-3 < x < 0	0 < x < 4	x > 4
x – 4	-	-	-	+
x + 3	-	+	+	+
Numerator	+	-	-	+
x	-	-	+	+
Quotient	-	+	-	+

$\therefore\ x-1<\frac{12}{x}$ when x < -3 or 0 < x < 4.



Ex. Solve $\frac{x+4}{x-2} \geq \frac{x-1}{x+5}$.

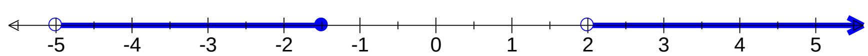
$$\frac{x+4}{x-2} - \frac{x-1}{x+5} \geq 0$$
$$\frac{(x+5)(x+4)-(x-2)(x-1)}{(x-2)(x+5)} \geq 0$$
$$\frac{x^2+9x+20-\left(x^2-3x+2\right)}{(x-2)(x+5)} \geq 0$$
$$\frac{12x+18}{(x-2)(x+5)} \geq 0$$

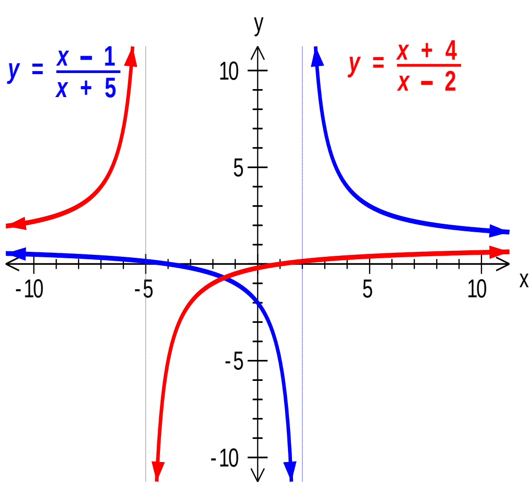
The rational expression is equal to 0 when $12x + 18 = 0$ or $x = -\frac{3}{2}$ so that must be included in the solution set.

The root of the numerator is $-\frac{3}{2}$. The roots of the denominator (VAs) are 2 and -5.

	$x < -5$	$-5 < x < -\frac{3}{2}$	$-\frac{3}{2} < x < 2$	$x > 2$
12x + 18	-	-	+	+
x - 2	-	-	-	+
x + 5	-	+	+	+
denominator	+	-	-	+
Quotient	-	+	-	+

$\frac{x+4}{x-2} \geq \frac{x-1}{x+5}$ when $-5 < x \leq -\frac{3}{2}$ or $x > 2$





HW: Pg. 295 #1, 3ab, 5cf, 7a

MHF4U – Unit 4

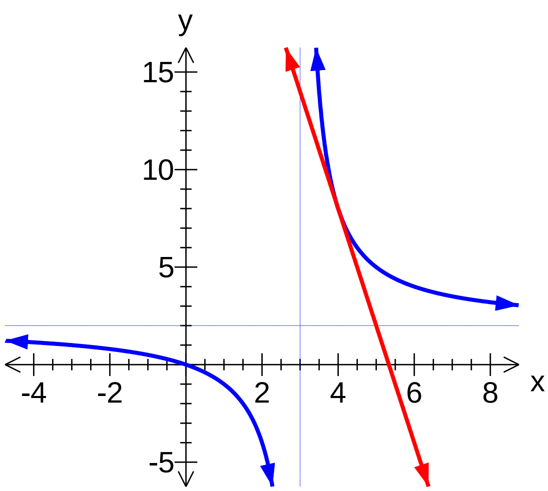
Lesson 6: Rates of Change in Rational Functions

Recall:

- The average rate of change of a function *f(x)* over the interval $x_1 \le x \le x_2$ if $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$.
  Graphically this is the slope of the secant through points (x_1, y_1) and (x_2, y_2) on $y = f(x)$.

- The instantaneous rate of change of a function *f(x)* at $x = a$ can be approximated using $\dfrac{f(a+h)-f(a)}{h}$ for very small values of *h*. Graphically this is the slope of the tangent through the point $(a, f(a))$ on $y = f(x)$.

Ex. Estimate the slope of the tangent to the graph of $f(x)=\dfrac{2x}{x-3}$ at the point where $x = 4$.



$$
\begin{aligned}
\text{Inst. RoC} &= \frac{f(a+h)-f(a)}{h}\text{, where } h = 0.001\\
&= \frac{f(4+0.001)-f(4)}{0.001}\\
&= \frac{f(4.001)-f(4)}{0.001}\\
&= \frac{\left[\frac{2(4.001)}{4.001-3}\right]-\left[\frac{2(4)}{4-3}\right]}{0.001}\\
&= \frac{7.99400599401-8}{0.001}\\
&= -5.994005994\\
&\doteq -6
\end{aligned}
$$

The slope of the tangent at $x = 4$ is approximately -6.

Can you find the slope of the tangent at a vertical asymptote? Horizontal asymptote? Hole?

Ex. The radius of a circular juice blot on a piece of paper towel t seconds after it was first seen is modeled by $r(t)=\frac{1+2t}{1+t}$, where r is measured in cm. Calculate the rate of increase of the blot after 1 second.

$$\begin{aligned}
m_{\tan} &= \frac{r(t+h)-r(t)}{h} \\
&= \frac{\left[\frac{1+2(t+h)}{1+(t+h)}\right]-\left[\frac{1+2t}{1+t}\right]}{h} \\
m_{\tan}\big|_{t=1} &= \frac{\left[\frac{1+2(1+0.001)}{1+(1+0.001)}\right]-\left[\frac{1+2(1)}{1+(1)}\right]}{0.001} \\
&= \frac{1.50024987506-1.5}{0.001} \\
&= 0.2498750625 \\
&\doteq 0.25
\end{aligned}$$

After 1 second, the blot is increasing in radius by 0.25 cm/second.

HW: Pg. 304 #6ab, 7, 10